Four channel Wigner-Smith matrix formalism applied to the scattering by a fluid layer embedded in semi infinite solids

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Abstract :

The acoustic scattering by a fluid slab between two semi infinite solid media is revisited from the point of view of a four channel resonant scattering formalism. For a plane and monochromatic longitudinal ($P=L$) or transversal ($P=T$) wave incident from each solid, the reflection coefficients r^{PQ} and transmission coefficients t^{PQ} by the fluid layer ($Q=L$, or T represents the polarization of the scattered waves) are the components of a 4x4 symmetric, unitary scattering matrix $\mathbf{S}$ ($\mathbf{S}^{\dagger}\mathbf{S}=\mathbf{S}\mathbf{S}^{\dagger}=\mathbf{I}$). In the particular case of identical solids, it is shown that $\mathbf{S}$ has to be written as the product of 2 unitary scattering matrices: $\mathbf{S}^{(b)}$ - corresponding to the scattering by the vacuumed layer-, and $\mathbf{S}^{(*)}$- denoting the pure resonant part of $\mathbf{S}$-. The Wigner-Smith matrix $\mathbf{Q}_{x_i}=-j(\partial_{x_i}\mathbf{S})\mathbf{S}^{\dagger}$ is analyzed (∂_{x_i} is the partial derivative with respect to a given input parameter $x_{i=1,..,4}=f, c_L, c_T, c_F$), this formalism being the multichannel extension of Phase Gradient Method.

Résumé :

L'étude de la diffusion acoustique par un slab fluide entre deux solides homogènes est réexaminée dans le cadre du formalisme de la matrice $\mathbf{S}$ de diffusion résonnante à 4 canaux. Considérant une onde incidente plane monochromatique longitudinale ($P=L$) ou transversale ($P=T$), la matrice de diffusion $\mathbf{S}$ (4x4) symétrique unitaire est élaborée à partir des coefficients de réflexion r^{PQ} et de transmission coefficients t^{PQ} par la couche fluide ($Q=L$, ou T représente la polarisation des ondes diffusées). Dans le cas particulier de solides identiques, on montre que $\mathbf{S}$ s'écrit comme produit de 2 matrices de diffusion unitaires: $\mathbf{S}^{(b)}$ - matrice de diffusion de la couche vide-, and $\mathbf{S}^{(*)}$-contribution purement résonnante de $\mathbf{S}$-. Le formalisme de la matrice de Wigner-Smith $\mathbf{Q}_{x_i}=-j(\partial_{x_i}\mathbf{S})\mathbf{S}^{\dagger}$ est alors mis en place (∂_{x_i} est la dérivée partielle par rapport à l'un des paramètres d'entrée $x_{i=1,..,4}=f, c_L, c_T, c_F$), afin de généraliser la Méthode des Gradients de Phase à une situation de diffusion multicanal.

Keywords :

Multichannel Scattering ; Phase Gradient ; Wigner-Smith Matrices

1 Construction of the $\mathbf{S}$ matrix

The present work is intended to prepare studies on effective medium approximations of a slab region randomly filled with scatterers in a solid. Let consider the simple problem of the scattering by a fluid layer F_2 (thickness d, density ρ_F, phase velocity c_F) between two semi infinite solids S_1 and S_3 (density $\rho_{S,m}$, longitudinal and transversal phase velocities $c_{L,m}$ and

$c_{T,m}$, $m=1,3$) sketched in FIG. 1. As for a plane and monochromatic incident wave (frequency f, incidence angle θ_P) from each solid S_m ($m=1,3$), two elementary polarizations $P=L$, or T (L for longitudinal, T for transversal) are available, such a study requires the multichannel framework provided by the scattering matrix $\mathbf{S}$ formalism.

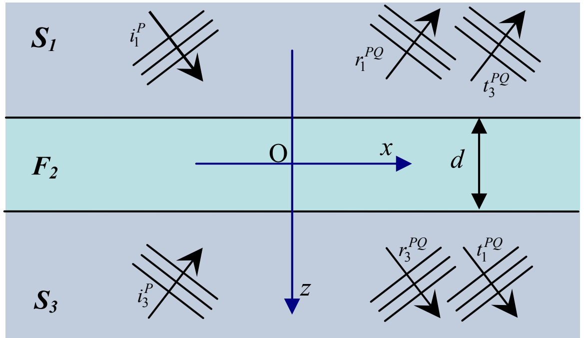

FIG. 1 – Geometrical configuration

Assuming the continuity of the normal and tangential stress components and the normal displacement component at the boundaries $z=\pm d/2$, the reflection r_m^{PQ} and transmission t_m^{PQ} coefficients of the structure are needed to build the $\mathbf{S}$ matrix. Their expression are (1)

$$r_m^{LL}=\left[(C_{A,m}^- + i\tau_m)(C_{S,n}^+ - i\tau_n)+(C_{A,n}^+ + i\tau_n)(C_{S,m}^- - i\tau_m)\right]/D,$$

$$r_m^{LT}=-4k_x k_{zL,m}(2k_x^2-k_{T,m}^2)\left[T(C_{S,n}^+ - i\tau_n)+C(C_{A,n}^+ + i\tau_n)\right]/D,\ r_m^{TL}=(k_{zT,m}/k_{zL,m})r_m^{LT},$$

$$r_m^{TT}=\left[(C_{A,m}^- - i\tau_m)(C_{S,n}^+ - i\tau_n)+(C_{A,n}^+ + i\tau_n)(C_{S,m}^- + i\tau_m)\right]/D,$$

$$t_m^{LL}=(\alpha_m/\alpha_n)\left[(C_{A,n}^- + i\tau_n)(C_{S,n}^+ - i\tau_n)-(C_{A,n}^+ + i\tau_n)(C_{S,n}^- - i\tau_n)\right]/D,$$

$$t_m^{LT}=4k_x k_{zL,m}(k_{T,m}^2/k_{T,n}^2)(2k_x^2-k_{T,m}^2)\left[T(C_{S,n}^+ - i\tau_n)-C(C_{A,n}^+ + i\tau_n)\right]/D,$$

$$t_m^{TL}=4k_x k_{zL,m}k_{T,m}^2/(k_{zL,n}k_{T,n}^2)k_{zT,m}(2k_x^2-k_{T,m}^2)\left[T(C_{S,n}^+ - i\tau_n)-C(C_{A,n}^+ + i\tau_n)\right]/D,$$

$$t_m^{TT}=-(\alpha_n/\alpha_m)\left[(C_{A,n}^- - i\tau_n)(C_{S,n}^+ - i\tau_n)-(C_{A,n}^+ + i\tau_n)(C_{S,n}^- + i\tau_n)\right]/D,$$

where (2)

$$D=(C_{A,m}^+ + i\tau_m)(C_{S,n}^+ - i\tau_n)+(C_{A,n}^+ + i\tau_n)(C_{S,m}^+ - i\tau_m),$$

$$C_{A,m}^\pm = R_m^\pm T,\ C_{S,m}^\pm = R_m^\pm C,\ \tau_m=(\rho_F/\rho_S)(k_{zL,m}/k_{zF}),$$

$$R_m^\pm = 4k_x^2 k_{zL,m}k_{zT,m}\pm(2k_x^2-k_{T,m}^2)^2,\ T=C^{-1}=\tan(\pi f k_{zF} d),$$

$$\alpha_m=(-i)2k_{T,m}^2(k_{zL,m}/k_{zF})(2k_x^2-k_{T,m}^2).$$

The indexes $m=1,3$ and $n\neq m$ stand for the incidence solid medium and the second solid medium respectively. Denoting by $\theta_{L,m}$ and $\theta_{T,m}$ the longitudinal and transversal incidence angles respectively and by θ_F the refraction angle in the fluid, the projections of the involved normalized wave vectors ($k_{L,m}=1/c_{L,m}, k_{T,m}=1/c_{T,m}, k_F=1/c_F$) are $k_x=\sin\theta_{L,m}/c_{L,m}=\sin\theta_{T,m}/c_{T,m}=\sin\theta_F/c_F$ along the x-axis, $k_{zL,m}=\cos\theta_{L,m}/c_{L,m}$, $k_{zT,m}=\cos\theta_{T,m}/c_{T,m}$, and $k_{zF}=\cos\theta_F/c_F$ along the z-axis. In the symmetric case of two identical solid media surrounding the slab ($S_3\equiv S_1$), the four channel scattering matrix $\mathbf{S}$ built up from the conservation of the energy flow vector perpendicularly to the interfaces (Franklin *et al.* (2001)) is

$$\mathbf{S} = \begin{bmatrix} r_1^{LL} & t_1^{LL} & \beta r_1^{LT} & \beta t_1^{LT} \\ t_1^{LL} & r_1^{LL} & -\beta t_1^{LT} & -\beta r_1^{LT} \\ \beta t_1^{LT} & -\beta t_1^{LT} & -r_1^{TT} & -t_1^{TT} \\ \beta t_1^{LT} & -\beta r_1^{LT} & -t_1^{TT} & -r_1^{TT} \end{bmatrix} \tag{3}$$

with the normalization coefficient $\beta = (k_{zT,1}/k_{zL,1})^{1/2}$. $\mathbf{S}$ is a symmetric and unitary matrix ($\mathbf{S}^\dagger\mathbf{S} = \mathbf{S}\mathbf{S}^\dagger = \mathbf{I}$, where † is the adjoint operator). In the following, the study will be limited to the case of light fluids ($c_F < c_{T,1} < c_{L,1}$), and the different graphics will be plotted for a water slab F_2 (thickness $d = 5$ mm, density $\rho_F = 1000$ kg.m^{-3}, phase velocity $c_F = 1470$ m.s^{-1}) in aluminum solid S_1 (density $\rho_{S,1} = 2790$ kg.m^{-3}, longitudinal and transversal phase velocities $c_{L,1} = 6380$ m.s^{-1} and $c_{T,1} = 3100$ m.s^{-1}).

2 Analysis of the $\mathbf{S}$ matrix : resonant and background scattering.

The S matrix formalism offers a powerful tool to break apart scattering mechanisms into simple and understandable contributions. As the analytical determination of the $\mathbf{S}$ eigenvalues from the characteristic equation $\det(\mathbf{S} - \lambda\mathbf{I}) = 0$ is difficult, a way to obtain these quantities consists in the study of the (2×2)-subminors of the determinant $\det(\mathbf{T})$ of the transition matrix $\mathbf{T} = (\mathbf{S} - \mathbf{I})/(2j)$. This provides the remarkable identities written as follows :

$$(r_1^{LL} - 1)/\beta r_1^{LT} = -t_1^{LL}/\beta t_1^{LT} = -\beta r_1^{LT}/(r_1^{TT} + 1) = -\beta t_1^{LT}/t_1^{TT}. \tag{4}$$

The use of the previous relations enables to factorize easily the charactacteristic polynomial $\det(\mathbf{S} - \lambda\mathbf{I})$. Finally, the four unitary eigenvalues of $\mathbf{S}$ are $\lambda = \{S_S, S_A, 1, 1\}$ where

$$S_{S/A} = r_1^{LL} - r_1^{TT} \pm (t_1^{LL} + t_1^{TT}) = -(C_{S/A,1} \pm j\tau_1)/(C_{S/A,1} \mp j\tau_1), \tag{5}$$

The open eigenchannels -associated with the eigenvalues S_S and S_A - correspond respectively to symmetric-longitudinal and antisymmetric-longitudinal polarizations. The eigenchannels associated with the double eigenvalue equal to 1 are closed and should correspond respectively to symmetric-transversal and antisymmetric-transversal polarizations if these channels were open (*i.e.* in the case of a solid slab for example).

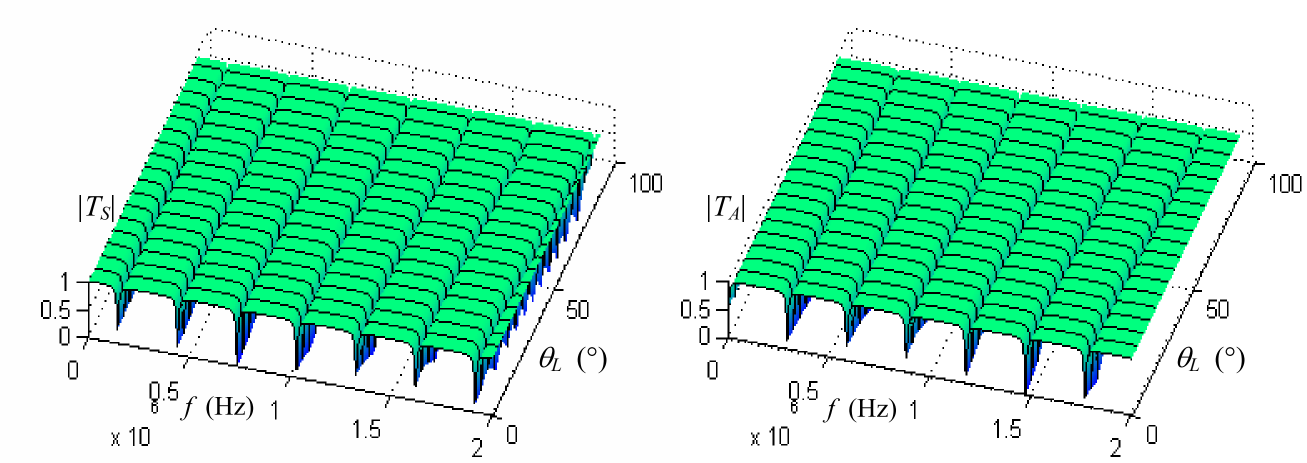

FIG. 2 – Plot in the (f , θ_L)-plane of the non null eigenvalue moduli of the transition matrix (left : |T_S|; right : |T_A|).

As a consequence, only two eigenvalues of the transition eigenmatrix $\mathbf{T}_{eig} = (\mathbf{S}_{eig} - \mathbf{I})/(2j)$ defined from the scattering eigenmatrix $\mathbf{S}_{eig} = \mathrm{diag}\{S_S, S_A, 1, 1\}$ are non null and denoted as $T_{S/A} = (S_{S/A} - 1)/(2j)$. The plots of their moduli in the (frequency, longitudinal incidence angle)-plane (f, θ_L) are given in FIG.2. They both exhibit antiresonances (in fact, resonances in counterphase with a constant transition amplitude component). Compared to other resonant scattering problems (Rembert *et al.* (2004)), this behavior can be easily explained by separating the whole scattering amplitudes as a combination of non resonant (or background) scattering terms and purely resonant scattering contributions. Mathematically speaking, the shape of the transition amplitudes is mainly due to the minus (-) sign that occurs in front of the non unimodular eigenvalues of $\mathbf{S}$ in Eq. (5). Indeed, when plotting the transition amplitudes $T^{(*)}_{S/A} = (-S_{S/A} - 1)/(2j)$ instead of $T_{S/A}$, one obtains a pure Breit-Wigner resonance spectrum (see FIG. 3). Indeed, at a given incidence angle θ_L, a first order expansion of $T^{(*)}_{S/A}$ shows that each resonance frequency is close to a root of $C_{S/A,1} = 0$ and each resonance half-width is nearly equal to $\pm\tau_1 / (\partial C_{S/A,1} / \partial x)$. As a consequence, the scattering eigenmatrices $\mathbf{S}^{(b)}_{eig} = \mathrm{diag}\{-1, -1, 1, 1\}$ and $\mathbf{S}^{(*)}_{eig} = \mathrm{diag}\{-S_S, -S_A, 1, 1\}$ are defined so that

$$\mathbf{S}_{eig} = \mathbf{S}^{(b)}_{eig} \mathbf{S}^{(*)}_{eig} . \tag{7}$$

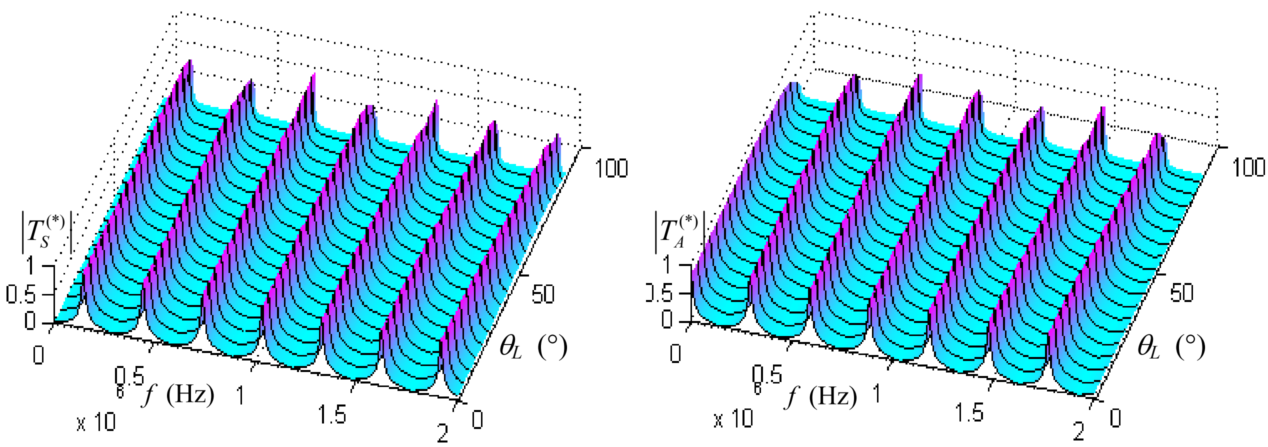

FIG. 3 – Plot in the (f , θ_L)-plane of the resonant eigentransition amplitudes (left : $\left|T^{(*)}_S\right|$; right : $\left|T^{(*)}_A\right|$).

In order to connect these eigenmatrices with the original scattering matrix $\mathbf{S}$, the calculation of the eigenvectors is straightforwardly processed. Once normalized, the real rotation matrix $\mathbf{R}$ involved in the change of basis relation, $\mathbf{S} = \mathbf{R}\mathbf{S}_{eig}\mathbf{R}^{\dagger}$, is

$$\mathbf{R} = \left[r_{ij} \right] = \frac{1}{\sqrt{2}\sqrt{1+\gamma^2}} \begin{bmatrix} 1 & 1 & \gamma & \gamma \\ -1 & 1 & \gamma & -\gamma \\ -\gamma & -\gamma & 1 & 1 \\ -\gamma & \gamma & -1 & 1 \end{bmatrix}, \tag{8}$$

where $\gamma = \sqrt{(R_1^+ + R_1^-)/(R_1^+ - R_1^-)}$. As they do not depend on the frequency, the components r_{11} and r_{13} of $\mathbf{R}$ are plotted versus the longitudinal incidence angle θ_L in FIG. 4.

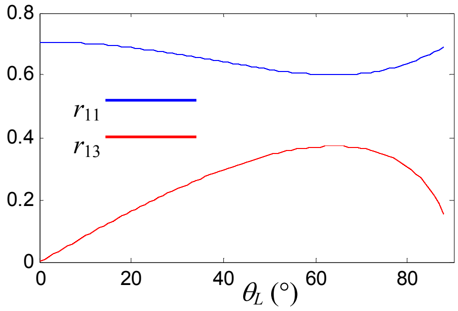

FIG. 4 – Plot of the r_{11} and r_{13} components of $\mathbf{R}$ versus θ_L

Introducing the change of basis relations $\mathbf{S}^{(*)} = \mathbf{R}\mathbf{S}^{(*)}_{eig}\mathbf{R}^t$ and $\mathbf{S}^{(b)} = \mathbf{R}\mathbf{S}^{(b)}_{eig}\mathbf{R}^t$, the scattering matrix $\mathbf{S}$ is factorized as the product $\mathbf{S} = \mathbf{S}^{(b)}\mathbf{S}^{(*)}$. $\mathbf{S}^{(*)}$ describes the pure resonant scattering of the fluid slab, and $\mathbf{S}^{(b)}$ the non resonant scattering of the slab. More precisely, it is demonstrated that $\mathbf{S}^{(b)}$ is nothing but the $\mathbf{S}$ matrix written in the particular case $\rho_F = 0$, *i.e.* the scattering by a vacuumed slab. In that case, the scattering coefficients related to the transmission terms vanish and the components of $\mathbf{S}^{(b)}$ only depend on the rotation matrix $\mathbf{R}$.

3 Wigner-Smith matrix

The time-delay or Wigner-Smith matrix $\mathbf{Q}_x$ is defined from the scattering matrix as

$$\mathbf{Q}_x = -j(\partial_x \mathbf{S})\mathbf{S}^{\dagger}. \tag{9}$$

Recalling that $\mathbf{S}^{-1} = \mathbf{S}^{\dagger}$, $\mathbf{Q}_x$ is the generalized logarithmic derivative of $\mathbf{S}$ with respect to a parameter x involved in the expression of the scattering matrix elements. This matrix is self-adjoint ($\mathbf{Q}_x = \mathbf{Q}_x^{\dagger}$) and its eigenvalues are pure phase partial derivatives with respect to x, connected with the eigenphase derivatives of $\mathbf{S}$ (Franklin *et al.* (2006)). $\mathbf{Q}_x$ is the multichannel generalization of the Phase Gradient Method used to investigate the resonant phase of scattering coefficients (Lenoir, *et al.* 2003). Considering the particular set of parameters $x_{i=1,..,4} = (f, c_F, c_{L,1}, c_{T,1})$, it is shown, as in the cases of one channel and two channel scattering (Lenoir, *et al.* 2003, Franklin *et al.* 2006), that the remarkable equation

$$\sum_{i=1}^{4} x_i \mathbf{Q}_{x_i} = 0, \tag{10}$$

is still verified. When written at a resonance frequency of the scatterer, this relation represents a reactive power balance. Another interest of the $\mathbf{Q}_x$ formalism lies obviously in the linearization of each possible factorization of $\mathbf{S}$. Inserting the change of basis relation $\mathbf{S} = \mathbf{R}\mathbf{S}_{eig}\mathbf{R}^{\dagger}$ in definition (9) gives

$$\mathbf{Q}_x = \mathbf{Q}_x^{(*)} + \mathbf{Q}_x^{(p)}. \tag{11}$$

In the previous relation, $\mathbf{Q}_x^{(*)} = -j\mathbf{R}(\partial_x S^{(*)}_{eig})S^{(*)\dagger}_{eig}\mathbf{R}^t$ is the purely resonant part of $\mathbf{Q}_x$ related to $\mathbf{S}^{(*)}$: its eigenmatrix is $(\partial_x \mathbf{S}^{(*)}_{eig})\mathbf{S}^{(*)\dagger}_{eig} = \mathrm{diag}\left\{-2\partial_x\delta_S^{(*)}, -2\partial_x\delta_A^{(*)}, 0, 0\right\}$, where $-\delta^{(*)}_{S/A}$ are the half resonant eigenphases of $\mathbf{S}$ (*i.e.* $S_{S/A} = -\exp(-j2\delta^{(*)}_{S/A})$), the trace of $\mathbf{Q}_x^{(*)}$ is a pure Breit-Wigner resonance spectrum (each resonance peak height are twice the inverse of their width). $\mathbf{Q}_x^{(p)}$ is the potential contribution of $\mathbf{Q}_x$ but slightly different from the background scattering due to $\mathbf{S}^{(b)}$. Indeed, its eigenmatrix $\mathbf{Q}^{(p)}_{x,eig}$ depends on the half resonant eigenphases :

$$\mathbf{Q}^{(p)}_{x,eig} = \mathrm{diag}\{\cos(\delta^{(*)}_S), \cos(\delta^{(*)}_A), -\cos(\delta^{(*)}_A), -\cos(\delta^{(*)}_S)\} \times \partial_x \gamma / (1+\gamma^2)\,. \quad (12)$$

$\mathbf{Q}^{(p)}_x$ is a null trace matrix, proving that $\mathrm{tr}(\mathbf{Q}_x) = \mathrm{tr}(\mathbf{Q}^{(*)}_x)$ contains the whole resonant information about the scatterer. In the particular case $x = f$, $\mathbf{Q}^{(p)}_x$ is null (γ is independent of f) and then, $\mathbf{Q}_f = \mathbf{Q}^{(*)}_f$. In Fig. 5, the first symmetric eigenvalues of $\mathbf{Q}^{(*)}_{c_F}$ and $\mathbf{Q}^{(p)}_{c_F}$ are plotted in the (f,θ_L)-plane : their respective amplitudes clearly indicate that the role of the potential scattering is negligible compared to the resonant scattering in $\mathbf{Q}_{c_F}$.

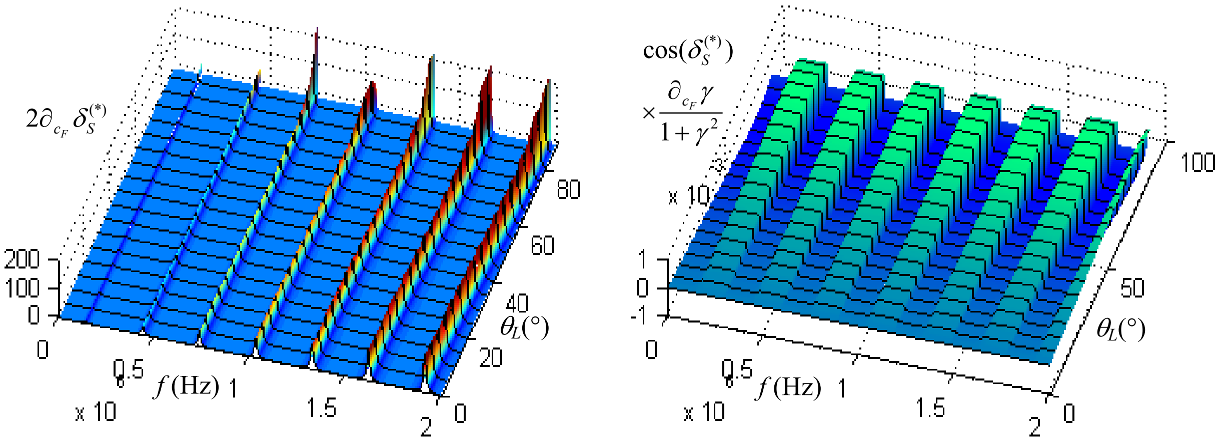

FIG. 5 – Plots in the (f,θ_L)-plane of the symmetric eigenvalues of $\mathbf{Q}^{(*)}_{c_F}$ (left), and $\mathbf{Q}^{(p)}_{c_F}$ (right).

5 Conclusive remarks

The Wigner-Smith matrix **Q** formalism is a natural filtering tool to isolate resonant contributions in a many scattering channel situation. This ability concerns, not only the frequency variable, but all the indepent input parameters of the problem. The **S** matrix formalism is used as a "*preprocessing engine"* in order to exploit fully the **Q** matrix properties.

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